-decays in Order $\alpha_s^4$ and Higher

Konstantin Chetyrkin
KIT (Karlsruhe Institute of Technology) & INR, Moscow

with P. Baikov (MSU) and J. H. Kühn (KIT)

• $O(\alpha_s^4)$ term in $R(s)$, $R_\tau$ and its structure

• implications for precise determination of $\alpha_s$ from $R_\tau$

• the WAR of strategies: CIPT versus FOPT and what one could do to reach a peaceful agreement/outcome?

• an ”improved” CIPT

• a question of scheme (not $\mu$!) dependence and scheme-independent treatment

• How reliable is the $O(\alpha_s^4)$ result per se?

• Conclusions
$\tau$ decays probe the correlator of the charged weak currents in an interesting region of energies just above 1 GeV

\[\Rightarrow\]

strong dependence on $\alpha_s$ and (for Cabibbo-suppressed part) on $m_s$

\[\Rightarrow\]

good for finding $\alpha_s$ and $m_s$

\[\Rightarrow\]

$\alpha_s$ is not very small $\Rightarrow$ higher order QCD terms are important $\Rightarrow$

they should be computed and understood
Theoretical Framework

\( R(s) \) is related (via unitarity) to the correlator of the bilinear quark currents:

\[
R(s) \approx \Im \Pi(s - i\delta)
\]

\[
3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx} \langle 0 | T[j_\mu^v(x)j_\nu^\mu(0)] | 0 \rangle dx
\]

To conveniently sum the RG-logs one uses the Adler function:

\[
D(Q^2) = Q^2 \frac{d}{dQ^2} \Pi(q^2) = Q^2 \int \frac{R(s)}{(s + Q^2)^2} ds
\]

or \((a_s \equiv \alpha_s/\pi)\)

\[
R(s) = \frac{1}{2\pi i} \int_{-s-i\delta}^{-s+i\delta} dQ^2 D(Q^2) = D(s) - \pi^2 \frac{\beta_0^2 d_0}{3} a_s^3 + \ldots
\]
R(s) at five loops is contributed by \( \approx 17 \cdot 10^3 \) of nonabelian or/and non-quenched diagrams like

\[ + \]

as well as 2671 purely abelian quenched diagrams like

\[ + \]
massless props \hspace{1cm} \leftrightarrow \hspace{1cm} \text{simplicity:} \\
5\text{-loop } R(s) \text{ is reducible}^* \\
to 4\text{-loop massless propagators } (\equiv \text{p-integrals}) \\
\hspace{1cm} \leftarrow \hspace{1cm} \hspace{1cm} \\
\text{main object to compute}
coefficient functions in front of master integrals depend on $D$ in simple way:

$$C^\alpha(D) = \frac{P^n(D)}{Q^m(D)} \xrightarrow{D \to \infty} \sum_k C_k^\alpha (1/D)^k$$

The terms in the $1/D$ expansion expressible (with the use of the Baikov’s representation) through simple Gaussian integrals

sufficiently many terms in $1/D$ and $C_k^\alpha \longrightarrow C^\alpha(D)$

computing time and required resources: could be huge (the price for full automatization); to cope with it we use parallel FORM / Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . .)

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All relevant Master Integrals solved analytically (2004) 
(method: “glue and cut” (Chetyrkin, Tkachov, (1981)) + BAICER )
\[ d_4 = n_f^3 \left[ -\frac{6131}{5832} + \frac{203}{324} \zeta_3 + \frac{5}{18} \zeta_5 \right] \text{ ("renormalon" chain /M. Beneke 1993/)} \\
+ n_f^2 \left[ \frac{1045381}{15552} - \frac{40655}{864} \zeta_3 + \frac{5}{6} \zeta_3^2 - \frac{260}{27} \zeta_5 \right] \text{ /Baikov, Kühn, K.Ch. (2002)/} \\
+ n_f \left[ \frac{13044007}{10368} \frac{12205}{12} \zeta_3 - 55 \zeta_3^2 + \frac{29675}{432} \zeta_5 + \frac{665}{72} \zeta_7 \right] \\
+ \left[ \frac{144939499}{20736} - \frac{5693495}{864} \zeta_3 + \frac{5445}{8} \zeta_3^2 + \frac{65945}{288} \zeta_5 - \frac{7315}{48} \zeta_7 \right] \\

Interesting features:

1. irrationals up to $\zeta_7$ (understandable from the structure of the masters)

2. no $\zeta_4$ and/or $\zeta_6$ (expected but mysterious!)
Result for the very $R(s)$

\[ R = 1 + a_s + (1.9857 - 0.1152n_f) a_s^2 + (-6.6369 - 1.2001n_f - 0.0052n_f^2) a_s^3 + \]
\[ + (-156.61 + 18.77n_f - 0.7974n_f^2 + 0.02152n_f^3) a_s^4 \]

and after separating dynamical from kinematical terms:

\[ R = 1 + \ldots (18.24 - 24.88 + (0.086 - 0.091)n_f^2 + (-4.22 + 3.02)n_f^3) a_s^3 \]
\[ + ((135.8 - 292.4 + (-34.4 + 53.2)n_f + (1.88 - 2.67)n_f^2 + (-0.010 + 0.031)n_f^3) a_s^4 \]

Note: the $\pi^2$-dominance (Radyushkin, Pivovarov, Kataev, Shirkov, \ldots) is not well pronounced
even more pronounced for the scalar correlator: and with “kinematical” $\pi^2$ terms explicitlty separated and underlined:

$$\tilde{R} = 1 + 5.667a_s + a_s^2 [51.57 - 15.63 - n_f(1.907 - 0.548)]$$
$$+ a_s^3 [648.7 - 484.6 - n_f(63.74 - 37.97) + n_f^2(0.929 - 0.67)]$$
$$+ a_s^4 [9471. - 9431. - n_f(1454.3 - 1233.4) + n_f^2(54.78 - 45.10)$$
$$- n_f^3(0.454 - 0.433)]$$

remarkable mutual cancellations in all $n_f$ powers!!!

for $n_f = 3 \rightarrow a_s^4(5589 - 6126) = -536.8$

similar cancellations happen for $\alpha_s^4 m_a^2/s$ and $\alpha_s^4 n_f^2$ terms in R(s)
FAC/PMS predictions* versus exact results

\[ n_f = 3 : \]
\[ r_4^{\text{FAC/PMS}} = -129 \pm 16 \quad \longleftrightarrow \quad r_4^{\text{exact}} = -106.88 = 48.08 - 155 \]

\[ n_f = 4 : \]
\[ r_4^{\text{FAC/PMS}} = -112 \pm 30 \quad \longleftrightarrow \quad r_4^{\text{exact}} = -92.89 = 27.34 - 120.28 \]

\[ n_f = 5 : \]
\[ r_4^{\text{FAC/PMS}} = -97 \pm 44 \quad \longleftrightarrow \quad r_4^{\text{exact}} = -79.98 = 9.21 - 89.191 \]

* (Kataev, Starshenko (95); Baikov, K.Ch., Kühn (2002))
impact on $\alpha_s$ from $Z$-decays

\[ R(s) = D(s) - \pi^2 \beta_0^2 \left\{ \frac{d_1}{3} a_s^3 + \left( d_2 + \frac{5\beta_1}{6\beta_0} d_1 \right) a_s^4 \right\} \]

\[ \Rightarrow \delta \alpha_s(M_Z) = 0.0005 \]

\[ \alpha_s(M_Z)^{\text{NNNLO}} = 0.1190 \pm 0.0026 \]

The theory error gets less by a factor 5 – 10!
impact on $\alpha_s$ from $\tau$-decays

$$\frac{\Gamma(\tau \to h_{s=0}\nu)}{\Gamma(\tau \to l\nu\nu)} = |V_{ud}|^2 S_{EW}^3 (1 + \delta_P + \delta_{EW}^{\text{small}} + \delta_{NP}^{\text{small}} + 0.003 \pm 0.003)$$

$$R_\tau = 3.471 \pm 0.011$$

(Davier, H"ocker, Zhang; ALEPH, OPAL, CLEO, . . .)

$\delta_P = 0.1998 \pm 0.043 \text{ (exp) scale } \mu^2/M_{\tau}^2 = 0.4 - 2$

<table>
<thead>
<tr>
<th>$d_4$</th>
<th>$\alpha_s^{FO}(M_\tau)$</th>
<th>$\alpha_s^{CI}(M_\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>no $\alpha_s^4$</td>
<td>0.337 $\pm$ 0.004 $\pm$ 0.03</td>
<td>0.354 $\pm$ 0.006 $\pm$ 0.02</td>
</tr>
<tr>
<td>$d_4 = 25$</td>
<td>0.325 $\pm$ 0.004 $\pm$ 0.02</td>
<td>0.347 $\pm$ 0.006 $\pm$ 0.009</td>
</tr>
<tr>
<td>$d_4 = 49.08$</td>
<td>0.322 $\pm$ 0.004 $\pm$ 0.02</td>
<td>0.342 $\pm$ 0.005 $\pm$ 0.01</td>
</tr>
</tbody>
</table>

use mean value between FOPT and CIPT*

\[ \alpha_s(M_\tau) = 0.332 \pm 0.005_{\text{exp}} \pm 0.015_{\text{theo}} \]

four-loop running\(^1\) + four-loop matching at quark thresholds\(^2\)

\[(m_c(m_c) = 1.286(13) \text{ GeV}, \ m_b(m_b) = 4.164(25) \text{ GeV})\]

\[\alpha_s(M_Z) = 0.1202 \pm 0.0006_{\text{exp}} \pm 0.0018_{\text{theo}} \pm 0.0003_{\text{evol}}\]
\[= 0.1202 \pm 0.0019\]

consistent with \(\alpha_s\) from \(Z\)

\(\delta \alpha_s\) from \(\tau\) dominated by theory.
\(\delta \alpha_s\) from \(Z\) dominated by statistics.

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CIPT versus FOPT; what to do?

idea 1: improved CIPT: one should not necessarily start from the Adler function, as after differentiation wrt $Q^2$ some useful information disappears (is moved to higher orders); it is certainly possible to RG improve the very polarization operator and integrate it over contour; the work is in progress...

idea 2: $\overline{\text{MS}}$ is not given by the God, it may well be not most appropriate scheme for the tau-lepton energy scale, I found scheme invariant approach (based on the idea of effective charge) as most appealing /by Körner, Krajewski and Pivovarov (2001)/
How reliable are our results?

History of $R(s)$ teaches us to be cautious:

∼ 20 years ago the famous Andrei Kataev (with Sergei Gorishny and Sergei Larin) first produced a severely wrong result for the $\mathcal{O}(\alpha_s^3)$ term (corrected only by three years later!) in $R(s)$ and now he himself (rightfully!) rises a question* of correctness of the first $\mathcal{O}(\alpha_s^4)$ result

* A. L. Kataev,

Is it possible to check urgently the 5-loop analytical results for the $e^+e^-$-annihilation Adler function?

hep-ph
we do understand the problem since long:

”A golden rule well-known among multi-loop people says that a result of a multi-loop calculation can be trusted and considered as the result only if it is confirmed by an independent calculation preferably made by a different group and with the use of the general covariant gauge. Therefore, in view of the importance of the $\mathcal{O}(\alpha_s^3)$ correction for both theory and phenomenology it is necessary to check those by a really independent calculation.”

(cited from K. Ch., Corrections of order $a_s^3$ to $R_{\text{had}}$ in $pQCD$ with light gluinos, Phys. Lett. B391, p. 403 (1997))

Unfortunately, at the moment we are not aware about any independent team which are going or, at least, able to check our results . . .
an important non-trivial check has been (unintentionally!) done in a remarkable recent work by A. Hoang, V. Mateu and S. Mohammad Zebarjad entitled “Heavy Quark Vacuum Polarization Function at $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$”

Here an attempt was made to colect all available information about massive four-loop polarization function (about its threshold behaviour, as well as low-energy moments and high-energy asymptotics) in order to restore the whole function within the Pade approach. An important ingredients were recently computed two lowest physical moments (Maier, Maierhöfer, Marquard (2008))

They predicted/computed the high-energy behaviour of the polarization operator in the form:

$$\Pi(Q^2, m^2) = a + b\frac{m^2}{Q^2} + \text{trivial logs (were used as input!)}$$

with some approximate values for a and b. E.g.

$$b(\text{from Hoang et al.}) = -3.885 \pm 0.417$$

while our exact result reads: \textcolor{red}{-4.3333} which just hits the left margin!

For the coefficient a the situation is similar!
• complete results on $R^{SS}(s)$, $R^{VV}(s)$ and, thus, on $R_{T}$ at $\mathcal{O}(\alpha_s^4)$ order are available

• they are partially (and non-trivially!) checked in a completely independent way

• higher order terms in these (and others too!!) massless correlators display strong cancellations between kinematical ($\sim \pi^2$) and dynamical contributions (in the $\overline{\text{MS}}$-scheme)

• the proper understanding/treatment of even higher orders is crucial for filling the notorious gap between CIPT and FOPT strategies

• new information is needed (BEWARE: any honest calculations of $\mathcal{O}(\alpha_s^5)$ and higher contributions is out for question in foreseeable future . . . )
Our Suggestions:

- "improved" Contour Improved Approach might be of use as it adds new (missing within classical CIPT using the Adler function) higher order contributions

- the scheme independent approach by Körner, Krajewski and Pivovarov could play a role of "independent judge" in the argument

- do not panic: attend related talks by world-leading experts on the workshop