# $\mathcal{T}$-decays in Order $\alpha_s^4$ and Higher

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- **$\mathcal{O}(\alpha_s^4)$ term in $R(s)$, $R_\tau$ and its structure**

- **implications for precize determination of $\alpha_s$ from $R_\tau$**

- **the WAR of strategies: CIPT versus FOPT and what one could do to reach a peaceful agreement/outcome?**

- **an "improved" CIPT**

- **a question of *scheme* (not $\mu$!) dependence and scheme-independent treatment**

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- **How reliable is the $\mathcal{O}(\alpha_s^4)$ result per se?**

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- **Conclusions**

**$\tau$ decays probe the correlator of the charged weak currents in an interesting region of energies just above 1 GeV**

$\Downarrow$

**strong dependence on $\alpha_s$ and (for Cabibbo-suppressed part) on $m_s$**

$\Downarrow$

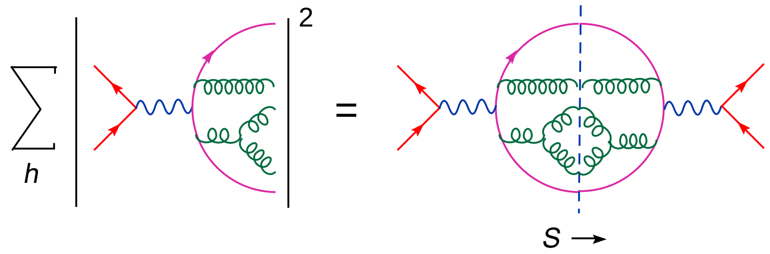
**good for finding $\alpha_s$ and $m_s$**

$\Downarrow$

**$\alpha_s$ is not very small $\Rightarrow$ higher order QCD terms are important $\Rightarrow$ they should be computed and understood**

# Theoretical Framework

R(s) is related (via unitarity) to the correlator of the bilinear quark currents:




$$R(s) \approx \Im\,\Pi(s - i\delta)$$

$$3Q^2\Pi(q^2 = -Q^2) = \int e^{iqx}\langle 0|T[\ j^v_\mu(x) j^v_\mu(0)\ ]|0\rangle dx$$

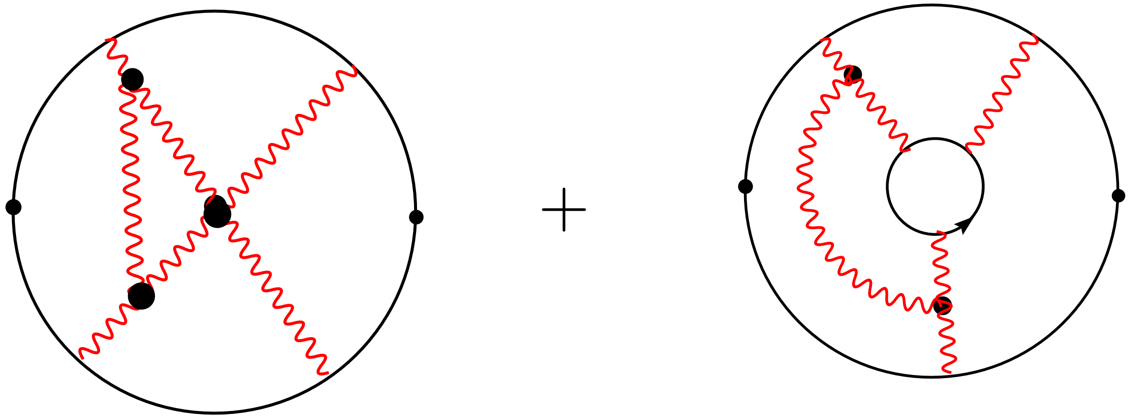
To conveniently sum the RG-logs one uses the Adler function:

$$D(Q^2) = Q^2 \frac{d}{d\,Q^2}\Pi(q^2) = Q^2 \int \frac{R(s)}{(s+Q^2)^2}\,ds$$

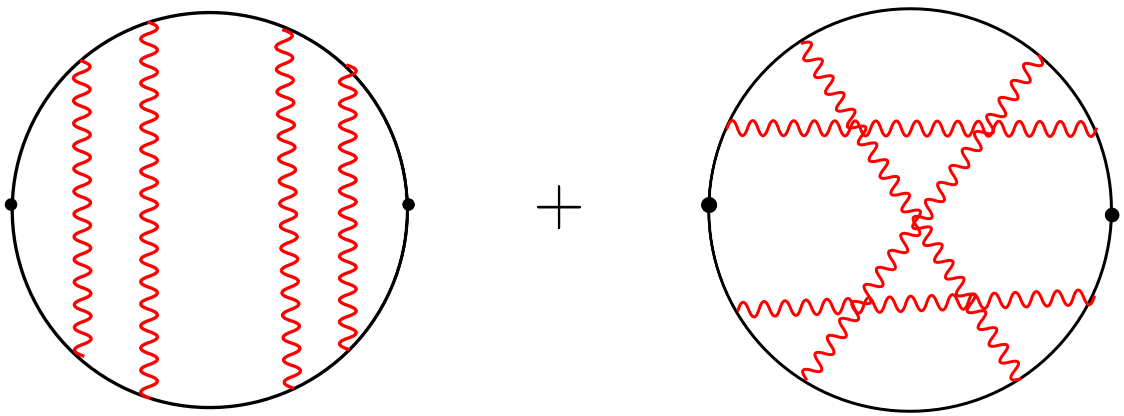
or $(a_s \equiv \alpha_s/\pi)$

$$R(s) = \frac{1}{2\pi i}\int_{-s-i\delta}^{-s+i\delta} dQ^2 \frac{D(Q^2)}{Q^2} = D(s) - \pi^2 \frac{\beta_0^2 d_0}{3} a_s^3 + \ldots$$

R(s) at five loops is contributed by $\approx 17 \cdot 10^3$ of nonabelian or/and non-quenched diagrams like



as well as 2671 purely abelian quenched diagrams like



**massless props $\longleftrightarrow$ simplicity:**

**5-loop R(s) is reducible$^{\star}$**

**to 4-loop massless propagators ($\equiv$ p-integrals)**

$\longleftarrow$

**main object to compute**

## reduction to Masters: $1/D$ expansion[1]

- coefficient functions in front of *master integrals* depend on $D$ in simple way:

$$C^\alpha(D) = \frac{P^n(D)}{Q^m(D)} \underset{D\to\infty}{=\!=\!=} \sum_k C_k^\alpha \;\; (1/D)^k$$

- The terms in the $\mathbf{1/D}$ expansion expressible (with the use of the Baikov's representation) through simple Gaussian integrals

- sufficiently many terms in $1/D$ and $C_k^\alpha \longrightarrow C^\alpha(D)$

- computing time and required resources: could be huge (the price for full automatization); to cope with it we use parallel FORM /Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – . . . )

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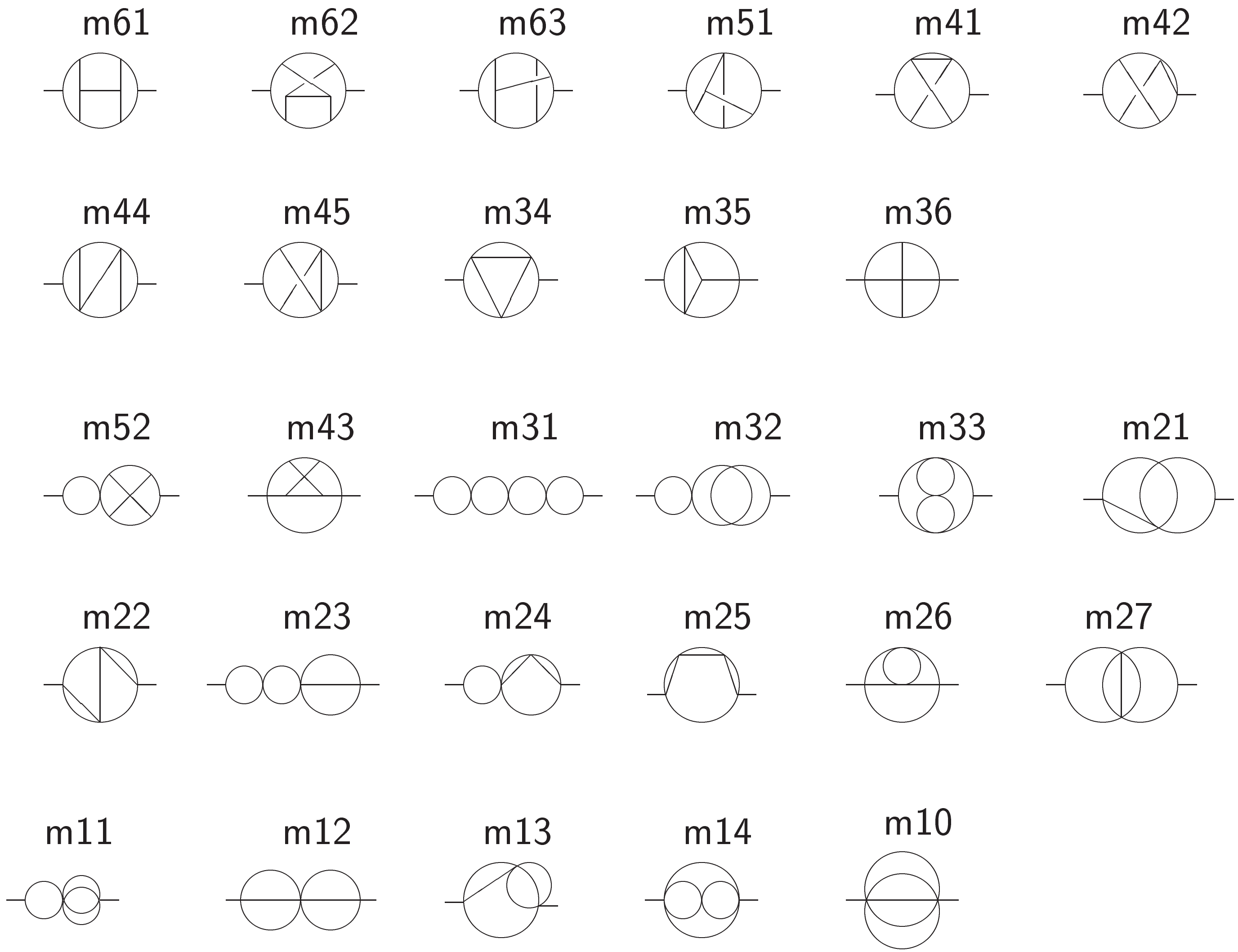
[1] Baikov, Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003

# All relevant Master Integrals solved analytically (2004)

(method: “glue and cut”(Chetyrkin, Tkachov, (1981)) + BAICER )



$$d_4 = n_f^3 \left[-\tfrac{6131}{5832} + \tfrac{203}{324}\zeta_3 + \tfrac{5}{18}\zeta_5\right]$$ ("renormalon" chain /M. Beneke 1993/)

$$+ n_f^2 \left[\tfrac{1045381}{15552} - \tfrac{40655}{864}\zeta_3 + \tfrac{5}{6}\zeta_3^2 - \tfrac{260}{27}\zeta_5\right]$$ /Baikov, Kühn, K.Ch. (2002)/

$$+ n_f \left[-\frac{13044007}{10368} + \frac{12205}{12}\zeta_3 - 55\,\zeta_3^2 + \frac{29675}{432}\zeta_5 + \frac{665}{72}\zeta_7\right]$$

$$+ \left[\frac{144939499}{20736} - \frac{5693495}{864}\zeta_3 + \frac{5445}{8}\zeta_3^2 + \frac{65945}{288}\zeta_5 - \frac{7315}{48}\zeta_7\right]$$

Interesting features:

1. irrationals up to $\zeta_7$ (understandable from the structure of the masters)

2. no $\zeta_4$ and/or $\zeta_6$ (expected but mysterious!)

# Result for the very R(s)

$$\boldsymbol{R = 1 + a_s + (1.9857 - 0.1152 n_f)\, a_s^2 + (-6.6369 - 1.2001 n_f - 0.0052 n_f^2)\, a_s^3 +}$$

$$\boldsymbol{+(-156.61 + 18.77\, n_f - 0.7974\, n_f^2 + 0.02152\, n_f^3)\, a_s^4}$$

and after separating <u>dynamical</u> from kinematical terms:

$$R = R = 1 + \ldots \left(\underline{18.24} - 24.88 + (\underline{0.086} - 0.091)\, n_f^2 + (\underline{-4.22} + 3.02)\, n_f^3\right)\; a_s^3$$

$$+ \left((\underline{135.8} - 292.4 + (\underline{-34.4} + 53.2)\, n_f + (\underline{1.88} - 2.67)\, n_f^2 + (\underline{-0.010} + 0.031)\, n_f^2\right)\; a_s^4$$

**note: the $\pi^2$-dominance (Radyushkin, Pivovarov, Kataev, Shirkov, $\ldots$) is not well pronounced**

even more pronounced for the scalar correlaror: and with "kinematical" $\pi^2$ terms explicitltly separated and underlined:

$$\widetilde{R} = 1 + 5.667 a_s + a_s^2 \left[51.57 - \underline{15.63} - n_f(1.907 - \underline{0.548})\right]$$
$$+ \quad a_s^3 \left[648.7 - \underline{484.6} - n_f(63.74 - \underline{37.97}) + n_f^2(0.929 - \underline{0.67})\right]$$
$$+ a_s^4 \left[9471. - \underline{9431.} - n_f(1454.3 - \underline{1233.4}) + n_f^2(54.78 - \underline{45.10})\right.$$
$$\left. - \quad n_f^3(0.454 - \underline{0.433})\right]$$

remarkable mutual cancellations in all $n_f$ powers!!!

for $n_f = 3 \longrightarrow a_s^4(5589 - 6126) = -536.8$

similar cancellations happen for ${\alpha_s}^4 m_a^2/s$ and ${\alpha_s}^4 n_f^2$ terms in R(s)

## FAC/PMS predictions* versus exact results

$n_f = 3:$

$$r_4^{\rm FAC/PMS} = -129 \pm 16 \quad \longleftrightarrow \quad r_4^{\rm exact} = -106.88 = \underline{48.08} - 155$$

$n_f = 4:$

$$r_4^{\rm FAC/PMS} = -112 \pm 30 \quad \longleftrightarrow \quad r_4^{\rm exact} = -92.89 = \underline{27.34} - 120.28$$

$n_f = 5:$

$$r_4^{\rm FAC/PMS} = -97 \pm 44 \quad \longleftrightarrow \quad r_4^{\rm exact} = -79.98 = \underline{9.21} - 89.191$$

* **(Kataev, Starshenko (95); Baikov, K.Ch., Kühn (2002))**

## impact on $\alpha_s$ from $Z$-decays

$$R(s) = D(s) - \pi^2 \beta_0^2 \left\{ \frac{d_1}{3} a_s^3 + \left( d_2 + \frac{5\beta_1}{6\beta_0} d_1 \right) a_s^4 \right\}$$

$$\Rightarrow \delta\alpha_s(M_Z) = 0.0005$$

$$\alpha_s(M_Z)^{\mathrm{NNNLO}} = 0.1190 \pm 0.0026$$

The theory error gets less by a factor 5 – 10!

# impact on $\alpha_s$ from $\tau$-decays

$$\frac{\Gamma(\tau \to h_{s=0}\nu)}{\Gamma(\tau \to l\overline{\nu}\nu)} = | V_{ud} |^2 S_{\text{EW}} 3 \, (1 + \delta_P + \underbrace{\delta_{\text{EW}}}_{\text{small}} + \underbrace{\delta_{\text{NP}}}_{0.003 \pm 0.003} )$$

$R_\tau = 3.471 \pm 0.011$

(Davier, Höcker, Zhang; ALEPH, OPAL, CLEO,. . . )

$\delta_P = 0.1998 \pm 0.043$ (exp) scale $\mu^2/M_\tau^2 = 0.4 - 2$

| | $\alpha_s^{FO}(M_\tau)$ | $\alpha_s^{CI}(M_\tau)$ |
|---|---|---|
| no $\alpha_s^4$ | $0.337 \pm 0.004 \pm 0.03$ | $0.354 \pm 0.006 \pm 0.02$ |
| $d_4 = 25$ | $0.325 \pm 0.004 \pm 0.02$ | $0.347 \pm 0.006 \pm 0.009$ |
| $d_4 = 49.08$ | $0.322 \pm 0.004 \pm 0.02$ | $0.342 \pm 0.005 \pm 0.01$ |

use mean value between FOPT and CIPT*

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*A.A. Pivovarov (1991,1992); F. Le Diberder and A. Pich (1992)/

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\rm exp} \pm 0.015_{\rm theo}$$

four-loop running[1] + four-loop matching at quark thresholds[2] ($m_c(m_c) = 1.286(13)$ GeV, $m_b(m_b) = 4.164(25)$ GeV)

$$\begin{aligned} \alpha_s(M_Z) &= 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm theo} \pm 0.0003_{\rm evol} \\ &= 0.1202 \pm 0.0019 \end{aligned}$$

consistent with $\alpha_s$ from $Z$

$\delta\alpha_s$ from $\tau$ dominated by theory.
$\delta\alpha_s$ from $Z$ dominated by statistics.

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[1] T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin (1997);M. Czakon (2005)
[2] Y. Schröder and M. Steinhauser (2006); K.G. Ch., J.H. Kühn, and C. Sturm (2006)

# CIPT versus FOPT; what to do?

**idea 1: improved CIPT: one should not necessarily start from the Adler function, as after differention wrt $Q^2$ some useful information disappears (is moved to higher orders); it is certanly possible to RG improve the very polarization operator and integrate it over contour; the work is in progress...**

**idea 2: $\overline{\text{MS}}$ is not given by the God, it may well be not most appropriate scheme for the tau-lepton energy scale, I found scheme invariant approach (based on the idea of effective charge) as most appealling /by Körner, Krajewski and Pivovarov (2001)/**

## How reliable are our results?

History of $R(s)$ teaches us to be cautious:

$\sim$ 20 years ago the famous Andrei Kataev (*with Sergei Gorishny and Sergei Larin* ) first produced a severly wrong result for the $\mathcal{O}(\alpha_s^3)$ term (corrected only by three years later!) in $R(s)$ and now he himself **(rightfully!)** rises a question* of correctness of the first $\mathcal{O}(\alpha_s^4)$ result

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* A. L. Kataev,

**Is it possible to check urgently the 5-loop analytical results for the $e^+e^-$-annihilation Adler function?**

hep-ph

we do understand the problem since long:

**"A golden rule well-known among multi-loop people says that a result of a multi-loop calculation can be trusted and considered as the result only if it is confirmed by an independent calculation preferably made by a different group and with the use of the general covariant gauge.**
**Therefore, in view of theimportance of the $\mathcal{O}(\alpha_s^3)$ correction for both theory and phenomenology it is necessary to check those by a really independent calculation."**

(cited from K. Ch., *Corrections of order $a_s^3$ to $R_{had}$ in pQCD with light gluinos,* Phys. Lett. **B391**, p. 403 (1997) )

Unfotunately, at the moment we are not aware about any independent team which are going or, at least, able to check our results ...

## an important non-trivial check has been (unintentionally!) done in a remarkable recent work by A. Hoang, V. Mateu and S. Mohammad Zebarjad entitled "Heavy Quark Vacuum Polarization Function at $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$"

Here an attempt was made to colect all availabe information about massive four-loop polarization function (about its threshold behaviour, as well as low-energy moments and high-energy asymptotics) in order to restore the whole function within the Pade approach. An important ingredients were recenly computed two lowest physical moments (Maier,Maierhöfer, Marquard (2008))

They *predicted/computed* the high-energy behaviour of the polarization operator in the form:

$$\Pi(Q^2, m^2) = a + b\frac{m^2}{Q^2} + \text{trivial logs (were used as input!)}$$

with some approximate values for a and b. E.g.

$b(from \;\; Hoang \;\; et \; al.) = -3.885 \pm 0.417$

while our **exact result** reads: **-4.3333** which just hits the left margin!

For the coefficient a the situation is similar!

# Summary

- complete results on $R^{SS}(s)$, $R^{VV}(s)$ and, thus, on $R_\tau$ at $\mathcal{O}({\alpha_s}^4)$ order are available
- they are partially (and non-trivially!) checked in a completely independent way
- higher order terms in these (and others too!!) massless correlators display strong cancellations between kinematical ($\sim \pi^2$) and dynamical contributions (in the $\overline{\text{MS}}$-scheme)
- the proper understanding/treatment of even higher orders is crucial for filling the notorious gap between CIPT and FOPT strategies
- new information is needed (BEWARE: any honest calculations of $\mathcal{O}({\alpha_s}^5)$ and higher contributions is out for question in foreseeable future . . . )

## Our Suggestions:

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- "improved" Contour Improved Approach might be of use as it adds new (*missing within classical CIPT using the Adler function*) higher order contributions

- the scheme independent approach by Körner, Krajewski and Pivovarov could play a role of "independent judge" in the argument

- do not panic: attend related talks by world-leading experts on the workshop