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Non-standard approach for hadronic tau decays

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Plan of Talk

– Introduction

– Model

– Result for R-related quantities

– About NP contributions

– Summary

Introduction

In comparing theoretical predictions with experimental data, it is important to connect measured quantities with 'simplest' theoretical objects. The hadronic correlator $\Pi(q^2)$ and the corresponding Adler function $D(Q^2)$, that appear in the process of e^+e^- annihilation into hadrons and the inclusive decay of the τ lepton, can play the role of these objects

$$\Pi_{\mu\nu}(q^2) = i \int d^4x\, e^{iqx}\, \left\langle 0 \left| T V_\mu(x) V_\nu(0)^+ \right| 0 \right\rangle$$

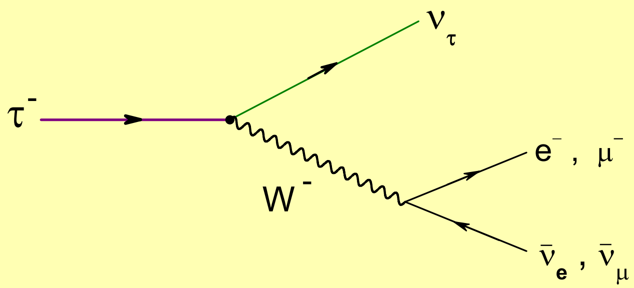
$$\propto (q_\mu q_\nu - g_{\mu\nu}\, q^2)\, \Pi(q^2) \qquad V^\mu_{ij} = \bar{\psi}_j \gamma^\mu \psi_i$$

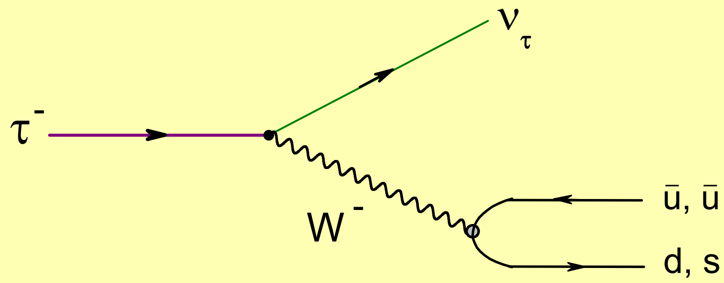
$$D(Q^2) \equiv -Q^2 \frac{d\,\Pi(-Q^2)}{d\,Q^2} \qquad Q^2 = -q^2 > 0$$

[in Euclidian (spacelike) region]

$$R(s) = \mathrm{Im}\Pi(s)/\pi$$

Hadronic τ decays are unique laboratory for low-energy QCD





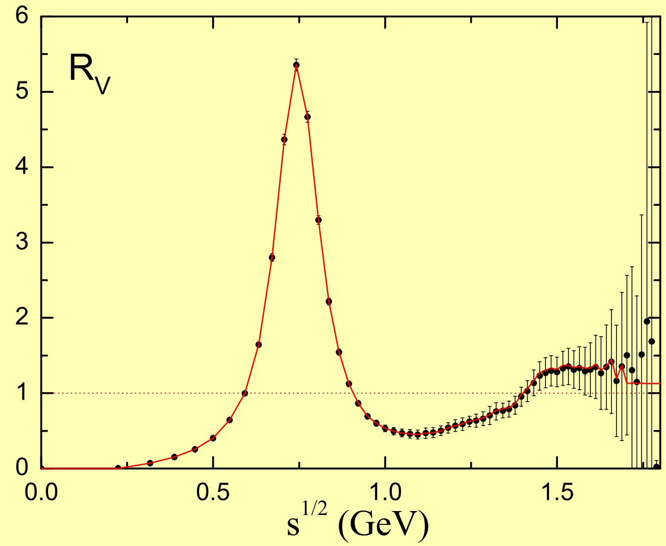
$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons}\ \nu_\tau)}{\Gamma(\tau^- \to \ell\ \bar{\nu}_\ell\ \nu_\tau)}$$

$$\cong N_C(|V_{ud}|^2 + |V_{us}|^2) \simeq 3$$

Vector channel

$$R_{\tau,V}^{\text{ALEPH}} = 1.775 \pm 0.017$$

$$R_{\tau,V}^{\text{OPAL}} = 1.764 \pm 0.016$$



- Ratio of hadronic to leptonic τ-decay widths in the vector channel

$$\boldsymbol{R}_\tau^V = R^{(0)} \int\limits_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) R(s)$$

- 'Light' Adler function (constructed from τ-decay data)

$$\boldsymbol{D}(Q^2) = Q^2 \int\limits_0^{\infty} ds \, \frac{R(s)}{(s+Q^2)^2}$$

- Smeared function

$$\boldsymbol{R}_\Delta(s) = \frac{\Delta}{\pi} \int\limits_0^{\infty} ds' \frac{R(s')}{(s-s')^2 + \Delta^2};$$

- Hadronic contribution to the anomalous magnetic moment of the muon

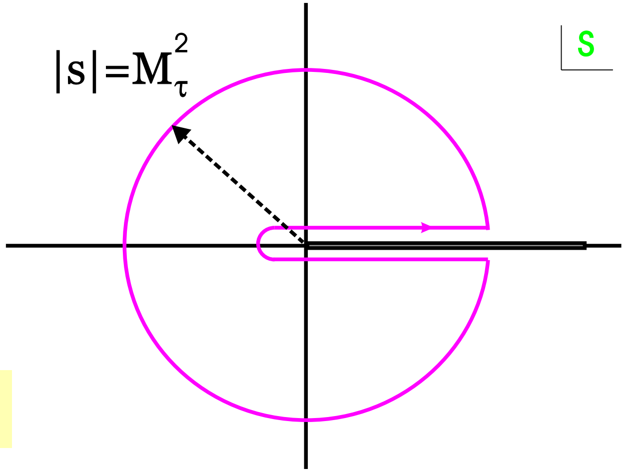
$$\boldsymbol{a}_\mu^{\rm had} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} K(s)\, R(s)$$

- and to the running of the fine structure constant

$$\boldsymbol{\Delta}\alpha_{\rm had}^{(5)}(s) = -\frac{\alpha(0)}{3\pi}\, s\, \mathcal{P} \int_0^\infty \frac{ds'}{s'} \frac{R(s')}{s'-s}.$$

A common feature of all these quantities and functions is that they are defined through the function $R(s)$ integrated with some other functions. By definition, all these quantities and functions include an infrared region as a part of the interval of integration and therefore, they cannot be directly calculated within perturbative QCD.

The initial integral is rewritten by using the Cauchy theorem in the form of a contour integral in the complex plane with the contour running around a circle with radius M_τ^2



$$R_\tau = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+2\frac{s}{M_\tau^2}\right) \; R(s)$$

$$= \frac{1}{2\pi \mathrm{i}} \oint\limits_{|z|=M_\tau^2} \frac{dz}{z}\left(1-\frac{z}{M_\tau^2}\right)^3\left(1+\frac{z}{M_\tau^2}\right) \; D(-z) \qquad (*)$$

$$= 3(|V_{ud}|^2+|V_{us}|^2)\, S_{\rm EW}(1+\delta_\tau) = R_\tau^{(0)}(1+\delta_\tau)$$

$|V_{ud}|$ and $|V_{us}|$ are CKM matrix elements, $S_{\rm EW}$ – electroweak factor

δ_τ – QCD contribution to R_τ-ratio

(I) If a calculation method maintains the correct analytic properties of the D-function, then both representations are equivalent.

Experimentally R_τ can be decomposed into the three contributions

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}$$

$R_{\tau,V}$ and $R_{\tau,A}$ are contributions coming from the non-strange hadronic decays associated with vector (V) and axial-vector (A) quark currents respectively, and $R_{\tau,S}$ contains strange decays (S).

$$R_{\tau,\ V/A}^{\mathrm{exp/theo}} = R_\tau^{(0)} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2\frac{s}{M_\tau^2}\right) R_{V/A}^{\mathrm{exp/theo}}$$

Within the PT with massless quarks $R_{\tau,V}$ and $R_{\tau,A}$ coincide with each other

$$\text{(II)} \qquad R_{\tau,V}^{\mathrm{PT}} = R_{\tau,A}^{\mathrm{PT}} = \frac{3}{2}|V_{ud}|^2(1 + \delta_\tau)$$

$$R_{\tau,V}^{\mathrm{exp}} \neq R_{\tau,A}^{\mathrm{exp}}$$

METHOD

The model based on

1) nonperturbative expansion method, variational perturbation theory (VPT)

VPT or in QCD the a-expansion method has been proposed by I. Solovtsov
PLB 327 and 340 (1994)

involves into analysis

2) nonperturbative character of the light quark masses is taken into account

3) summation of infinite number of threshold singularities.

[Milton and Solovtsov (2001)]

The method we used here is the non-perturbative approach based on the idea of variational perturbation theory which combines an optimization procedure of variational type with a regular method of calculating corrections. In the case of QCD the non-perturbative expansion parameter, a, obeys an equation whose solutions are always smaller than unity for any value of the original coupling constant. An important feature of this approach is the fact that for sufficiently small value of the running coupling the a-expansion reproduces the standard perturbative expansion, and, therefore, the perturbative high-energy physics is preserved. In moving to low energies, where ordinary perturbation theory breaks down ($\bar{\alpha}_s \simeq 1$), the parameter a remains small and we still stay within the region of applicability of the a-expansion method.

The method of VPT gives the following connection between the expansion parameter a and the original coupling constant

$$\lambda = \frac{g^2}{(4\pi)^2} = \frac{\alpha_s}{4\pi} = \frac{1}{C}\,\frac{a^2}{(1-a)^3}\,.$$

For all values of the running coupling $\lambda \geq 0$ the expansion parameter a obeys the inequality $0 \leq a < 1$. The positive parameter C plays the role of a variational parameter.

The renormalization group β-function in the leading order is

$$\beta_a(a) = \mu^2\,\frac{\partial\, a}{\partial\,\mu^2} = \frac{2\,\beta_0}{C}\,\frac{1}{f'(a)}\,,$$

where $\beta_0 = 11 - 2/3\,N_f$ is the first coefficient of the standard β-function in perturbative expansion, and N_f is the number of active quarks.

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48\,\ln a - \frac{18}{11}\,\frac{1}{1-a} + \frac{624}{121}\,\ln(1-a) + \frac{5184}{121}\,\ln(1+\frac{9}{2}\,a)$$

with an accuracy $O(a^3)$.

The momentum dependence of the running expansion parameter $a(Q^2)$ is finds as a solution of the transcendental equation

$$\ln \frac{Q^2}{Q_0^2} = \frac{C}{2\,\beta_0}\,[\,f(a) - f(a_0)\,]\,.$$

The function $f(a)$ is a *monotonous* in the interval $(0,1)$ and, therefore, for any values of Q^2, the equation has a unigue solution in the inyerval $0 < a(Q^2) < 1\,.$

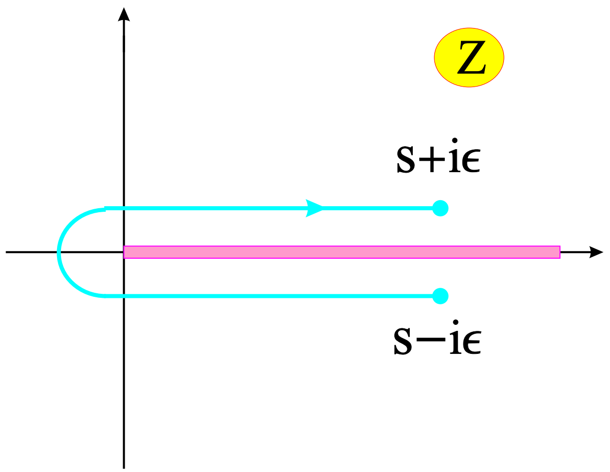
The parameter C can be defined, if one takes into account the Källén–Lehmann analyticity of the running parameter $a(Q^2)$.

In the massless case

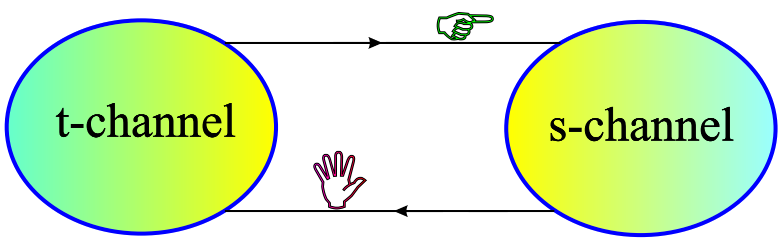
$$R(s) = \left[1 + r_0 \lambda_s^{\rm eff}(s)\right], \quad D(Q^2) = \left[1 + d_0 \lambda^{\rm eff}(Q^2)\right].$$

$$\lambda^{\rm eff}(q^2) = -q^2 \int_0^\infty \frac{d\,s}{(s-q^2)^2}\,\lambda_s^{\rm eff}(s),$$

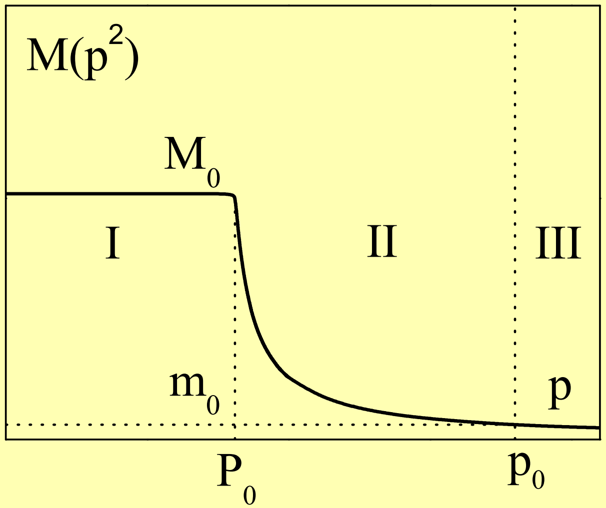
$$\lambda_s^{\rm eff}(s) = -\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\,\epsilon}^{s+i\epsilon} \frac{d\,z}{z}\,\lambda_t^{\rm eff}(z).$$



The VPT leads to a self-consistent definition of analytic continuation.



Model mass function



At small p^2 the function is rather smooth ($M_0 \simeq 260$ MeV). In the region $p^2 > 1 \div 2$ GeV the principle behavior is defined by PT. The parameters m_0 are taken from the known values of the running (current) masses at $p_b = 2$ GeV.

$f_\pi\ (\pi \to \mu\nu)$ $\qquad$ $f_\pi^{\exp} = 92.40 \pm 0.26$ МэВ.

Pagels-Stokar expression

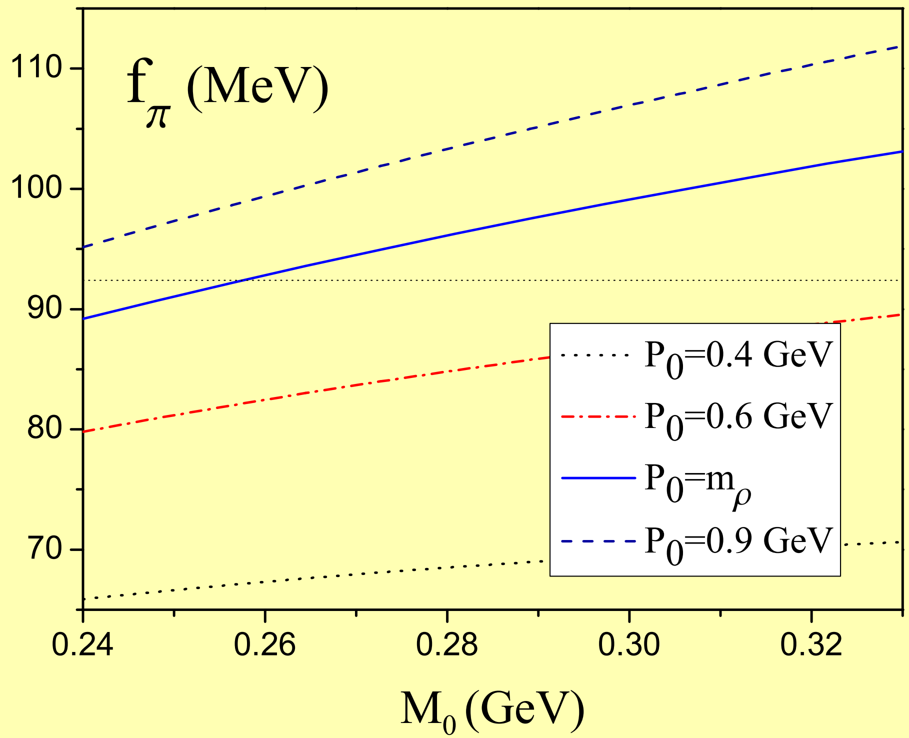
$$f_\pi^2 = \frac{3}{4\pi^2} \int\limits_0^\infty dp^2 \, \frac{p^2\, M(p^2)}{[p^2 + M^2(p^2)]^2} \left[M(p^2) - \frac{p^2}{2} \frac{dM(p^2)}{dp^2} \right]$$

$$f_\pi^2 = f_{\pi,I}^2 + f_{\pi,II}^2 + f_{\pi,III}^2.$$

$f_{\pi,I}^2$ – dominant

$f_{\pi,II}^2$ – near by 20%

$f_{\pi,III}^2$ – numerically small



So, additional information about f_π allows to get the new restrictions on parameters of mass function (P_0 close to value of mass ρ-meson).

The quark condensate $\langle \bar{q}q \rangle$

$$\langle \bar{q}q \rangle_a == -\frac{N_c}{4\pi^2} \int_0^{a^2} dp^2 \frac{p^2 \, M(p^2)}{p^2 + M^2(p^2)}$$

At the same parameters of mass function ($M_0 = 0.26$ GeV, $P_0 = m_\rho$, $m_0 = 4$ MeV and $p_0 = 2$ GeV), we get

$$\langle 0|\bar{q}q|0\rangle\big|_{p_0^2} = -(235\ \mathrm{MeV})^3 \Rightarrow \langle 0|\bar{q}q|0\rangle\big|_{\mu^2=1} = -(220\ \mathrm{MeV})^3$$

Phenomenological value is $\langle 0|\bar{q}q|0\rangle = -(240\ \mathrm{MeV})^3$

From the Gell-Mann–Oakes-Renner relation

$$\langle 0|\bar{q}q|0\rangle = -\frac{m_\pi^2 \, f_\pi^2}{2\, m_0} \Rightarrow \langle 0|\bar{q}q|0\rangle = -(210\ \mathrm{MeV})^3$$

To incorporate the quark mass effects one usually uses

$$\mathcal{R}_V(s) = T(v)\,\Theta(s - 4m^2)\ [1 + g(v)r(s)]\ ,$$

where

$$T(v) = \frac{v\,(3 - v^2)}{2}\,, \qquad v = \sqrt{1 - \frac{4m^2}{s}}\,,$$

and the function $g(v)$ in the Schwinger approximation is

$$g(v) = \frac{4\pi}{3}\left[\frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi}\right)\right]\,.$$

As $m \to 0$, $\quad \mathcal{R}(s)\ =\ 1\ +\ r(s)\ \ .$

Threshold singularities

Problem: Real expansion parameter is $\dfrac{\alpha}{v}$.

One cannot truncate the PT series in the threshold region.

For heavy quark system, one usually applies (well-known from QED) the Sommerfeld-Sakharov factor.

$$\mathcal{R}(s) \;\propto\; |\chi_{BS}(0)|^2 \;\Longrightarrow\; |\psi_{Sh}(0)|^2 \Longrightarrow S(v)\,.$$

A new relativistic factor in the case of QCD has been proposed by Milton & Solovtsov by using the quasipotential approach to quantum field theory [Logunov & Tavkhelidze'63] in the form sugested by Kadyshevsky.

Threshold S-factor relativistic generalization of the Sommerfeld-Sakharov factor with correct NR and UR limits [Milton and Solovtsov (2001)]

$$S(\chi) = \frac{X(\chi)}{1-\exp\left[-X(\chi)\right]}, \qquad X(\chi) = \frac{4\,\pi\,\alpha_s}{3\,\sinh\chi},$$

where χ is the rapidity which related to s by $2m\cosh\chi = \sqrt{s}$.

The function $X(\chi)$ in can be expressed in terms of $v = \sqrt{1-4m^2/s}$ as $X(\chi) = \pi\alpha\sqrt{1-v^2}/v$.

To take into account the threshold resummation factor, we modify R-function

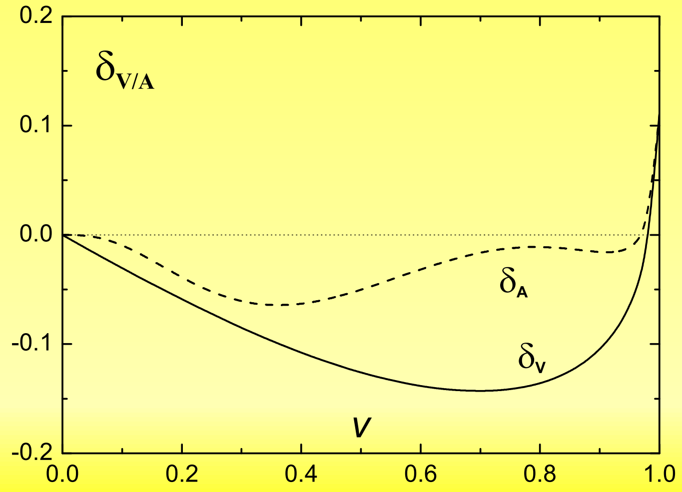
$$\mathcal{R}_f(s) = \Big[R_{0,f}(s) + R_{1,f}(s)\Big]\,\Theta(s-4m_f^2),$$

$$R_0(s) = T(v)\,S(\chi),$$

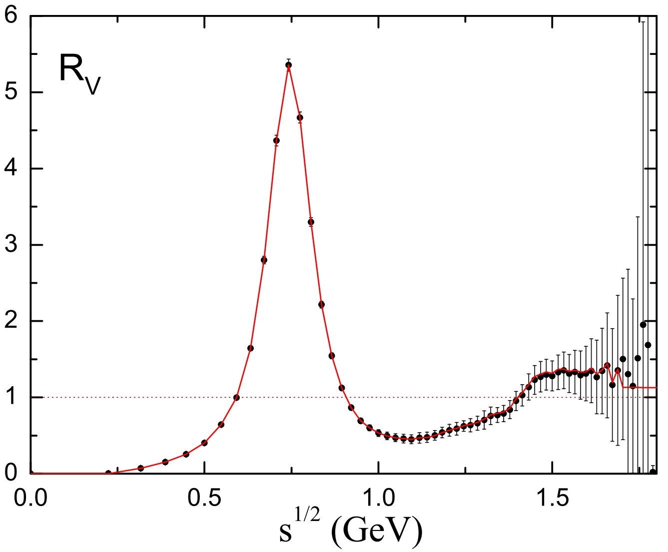
$$R_1(s) = T(v)\left[r_{\rm VPT}(s)\,g(v) - \frac{1}{2}X(\chi)\right].$$

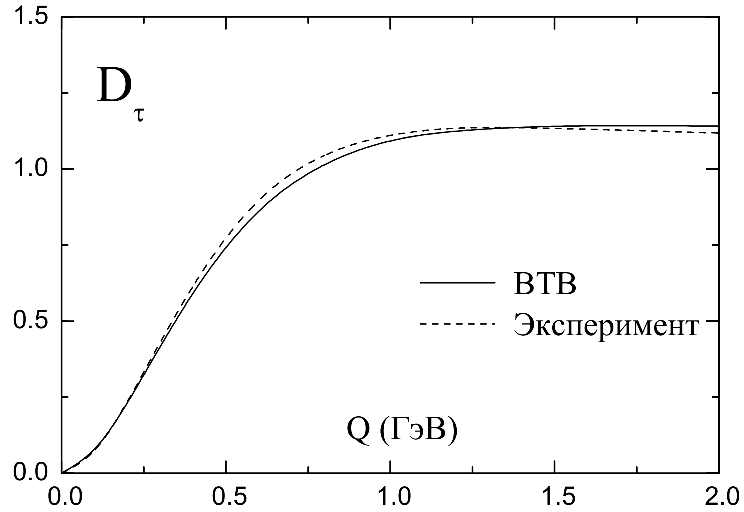
$\mathcal{R}(s) \Rightarrow 1 + \dfrac{\alpha}{\pi}$, $m \to 0$

$\delta_V \equiv R_1/R_0 \;\Rightarrow$



Vector channel in τ decay





$$R^{\text{exp}}_{\tau,V} = 1.787 \pm 0.013$$

$$R^{\text{VPT}}_{\tau,V} = 1.79 = R^{\text{exp},\text{centr}}_{\tau,V}$$

$$D_{\tau,V/A}(Q^2) = Q^2 \int\limits_0^\infty ds \, \frac{R_{\,V/A}(s)}{(s+Q^2)^2}$$

Axial-vector channel

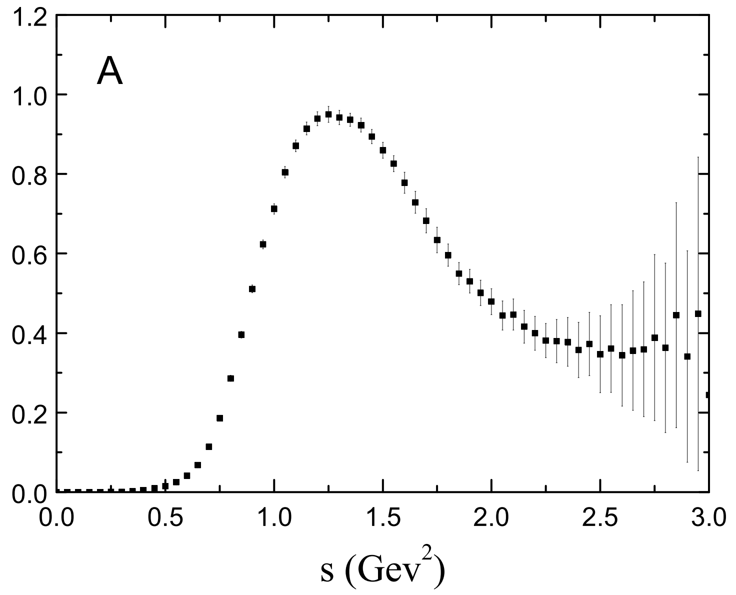
$R^{\mathrm{exp}}_{\tau,A} = 1.695 \pm 0.013$

$R_{\tau,A} \Rightarrow \quad R^{(1)}_{\tau,A} + R^{(0)}_{\tau,A}$

$R^{\mathrm{exp}}_{\tau,\pi} = 0.612 \pm 0.004$

$R^{\mathrm{exp}}_{\tau,A1} = 1.083 \pm 0.014$

$R^{\mathrm{VPT}}_{\tau,A} = 1.092$



The contribution to the imaginary part of the axial-vector correlator, $\mathrm{Im}\Pi^{(0)}$, is taken from the pion pole

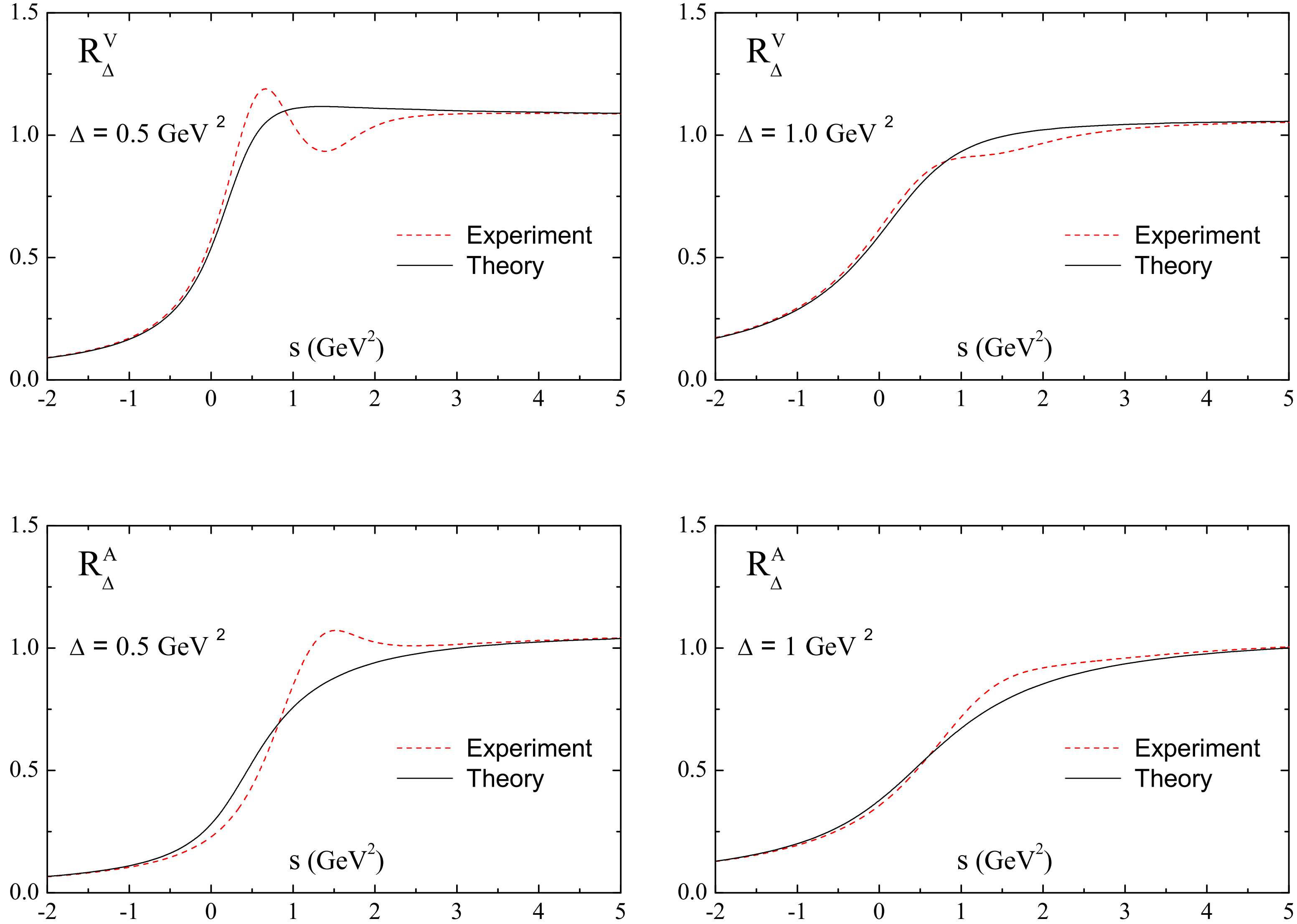
$$R_{\tau,\pi} = 3|V_{ud}|^2 S_{\mathrm{EW}} \frac{8\pi^2 f_\pi^2}{M_\tau^2} \left(1 - \frac{m_\pi}{M_\tau^2}\right)^2 \rightarrow 0.612 \pm 0.004$$

Smeared R_Δ-function

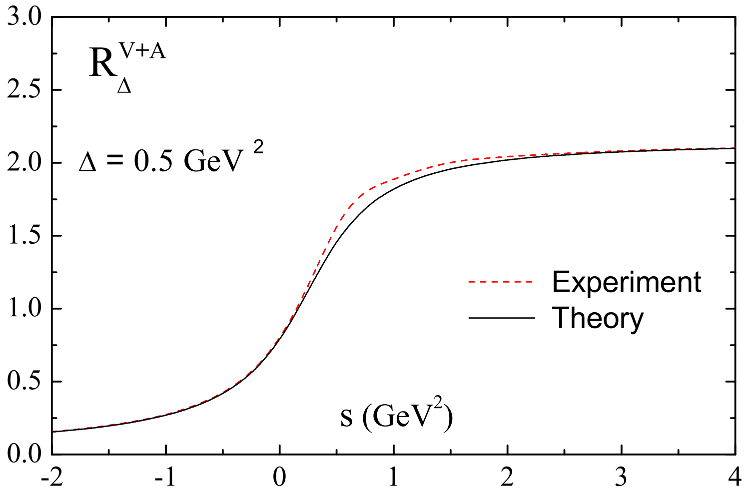
[Poggio, Quinn, Weinberg 1976]

$$R_\Delta(s) = \frac{1}{2\pi\, i} \left[\Pi(s + i\Delta) - \Pi(s - i\Delta) \right]$$

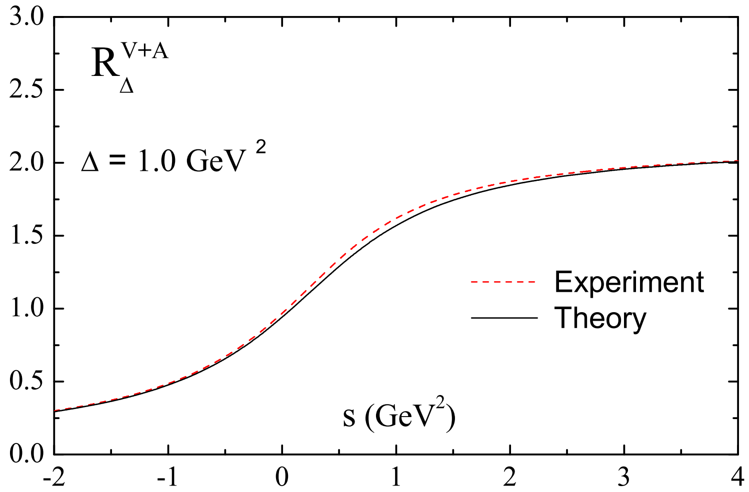
$$R_\Delta^{\mathrm{exp/theo}}(s) = \frac{\Delta}{\pi} \int_0^\infty ds' \, \frac{R_{V/A}^{\mathrm{exp/theo}}(s')}{(s-s')^2 + \Delta^2}$$



R_Δ^V
Δ = 0.5 GeV ^{2}
Experiment
Theory
s (GeV^2)
R_Δ^V
Δ = 1.0 GeV ^{2}
Experiment
Theory
s (GeV^2)
R_Δ^A
Δ = 0.5 GeV ^{2}
Experiment
Theory
s (GeV^2)
R_Δ^A
Δ = 1 GeV ^{2}
Experiment
Theory
s (GeV^2)



Vector + Axial-vector smeared function $R_\Delta^{V+A}(s)$ for $\Delta = 0.5$ GeV^2.



Vector + Axial-vector smeared function $R_\Delta^{V+A}(s)$ for $\Delta = 1.0$ GeV^2.

The method allows us to describe well smeared functions.

Let us emphasize that in the spacelike region ($s \lesssim 0$) there is an excellent agreement between data and theory.

Hadronic contributions to a_μ

$$a_\mu^{\rm had} = (702 \pm 16) \times 10^{-10}.$$

Other results for $a_\mu \times 10^{-10}$:

$696.3 \pm 6.2_{\rm exp} \pm 3.6_{\rm rad}$	e^+e^--based [Davier 2003]
$711.0 \pm 5.0_{\rm exp} \pm 0.8_{\rm rad} \pm 2.8_{SU(2)}$	τ-based [Davier 2003]
$693.4 \pm 5.3_{\rm exp} \pm 3.5_{\rm rad}$	e^+e^--based [Hocker 2004]
623.0 ± 40	Instanton liquid model [Dorokhov 2005]
698 ± 23	APT-approach [Milton, Solovtsov & S 2006]

Hadronic contributions to $\Delta\alpha$

$$\alpha(s) = \frac{\alpha(0)}{1 - \Delta\alpha_{\rm lept}(s) - \Delta\alpha^{(5)}_{\rm had}(s) - \Delta\alpha^{\rm top}_{\rm had}(s)}.$$

The leptonic part $\Delta\alpha_{\rm lept}(s)$ is known to the three loop level, $\Delta\alpha_{\rm lept}(M_Z^2) = 0.03149769$. It is conventional to separate the contribution $\Delta\alpha^{(5)}_{\rm had}(s)$ coming from the first five quark flavors. The contribution of the t-quark is estimated as $\Delta\alpha^{\rm top}_{\rm had}(M_Z^2) = -0.000070(05)$.

Our result

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = (279.9 \pm 4.0) \times 10^{-4}.$$

This value is to be compared with predictions extracted from a wide range of data describing $e^+e^- \to$ hadrons [Hagiwara *et al.* 2004]:

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = (275.5 \pm 1.9_{\rm exp} \pm 1.3_{\rm rad}) \times 10^{-4}.$$

Connection with Operator Product Expansion

$$\Pi_{\rm OPE}(-Q^2) = \Pi_{PT}(-Q^2) + \frac{\langle\mathcal{O}_2\rangle}{Q^2} + \frac{\langle\mathcal{O}_4\rangle}{Q^4} + \frac{\langle\mathcal{O}_6\rangle}{Q^6} + \cdots$$

$\mathcal{O}_{2n}$ are the local operators constructed from quark and gluon fields.

Within the standard QCD sum rules approach $\langle\mathcal{O}_2\rangle = 0$ (no a gauge invariant operator of dimension two)

$$\langle\mathcal{O}_4\rangle = \frac{\pi^2}{3}\langle 0|\frac{\alpha_s}{\pi}G^2|0\rangle + 4\pi^2\langle 0|m_u\bar{u}u + m_d\bar{d}d|0\rangle,$$

$$\langle\mathcal{O}_6\rangle = -4\pi^3\langle 0|\alpha_s\left[\bar{u}\gamma_\mu\gamma_5\lambda^a u - \bar{d}\gamma_\mu\gamma_5\lambda^a d\right]^2|0\rangle$$

$$-\frac{8\pi^3}{9}\langle 0|\left(\bar{u}\gamma_\mu\lambda^a u + \bar{d}\gamma_\mu\lambda^a d\right)\sum_{q=u,d,s}\bar{q}\gamma_\mu\lambda^a q|0\rangle.$$

Phenomenological values are

$$\langle\mathcal{O}_4\rangle_{\rm phen} \simeq 0.04\,{\rm GeV}^4, \quad \langle\mathcal{O}_6\rangle_{\rm phen} \simeq -0.06\,{\rm GeV}^6.$$

Residual condensates

$\Delta\Pi(Q^2) = \Pi_{\rm expt}(Q^2) - \Pi_{\rm theor}(Q^2)$ $\Leftarrow$ *measure of knowledge/ignorance*

Dispersion relation

$$\Delta\Pi(Q^2) = \int_0^\infty ds \frac{\Delta R(s)}{s+Q^2}, \qquad \Delta R(s) = R_{\rm expt}(s) - R_{\rm theor}(s)$$

If one is able to derive the function $R_{\rm theor}(s)$ exactly, in this case $\Delta R(s) = 0$.

Here

$$\Delta\Pi(Q^2) = \Pi_{\rm expt}(Q^2) - \Pi_{\rm theor}(Q^2) = \frac{\langle O_2\rangle}{Q^2} + \frac{\langle O_4\rangle}{Q^4} + \frac{\langle O_6\rangle}{Q^6} + \cdots$$

$\langle O_{2n}\rangle$ are residual condensates

The Borel sum rules

The Borel transform of $\Delta\Pi(Q^2)$ gives

$$\langle O_2\rangle + \frac{\langle O_4\rangle}{M^2} + \frac{\langle O_6\rangle}{2M^4} + \cdots = \mu_1(M^2)$$

The parameter s_0 is taken in such a way that for $s > s_0$ one can suppose the theoretical result for $R(s)$ reproduces experimental data with enough accuracy.

System of sum rules

$$\begin{aligned}
\langle O_2\rangle + \frac{\langle O_4\rangle}{M^2} + \frac{\langle O_6\rangle}{2M^4} &= \mu_1(M^2), \\
\langle O_4\rangle + \frac{\langle O_6\rangle}{M^2} &= -\mu_2(M^2), \\
\langle O_2\rangle + \frac{\langle O_4\rangle}{2M^2} + \frac{\langle O_6\rangle}{6M^4} &= \tilde{\mu}(M^2),
\end{aligned}$$

where

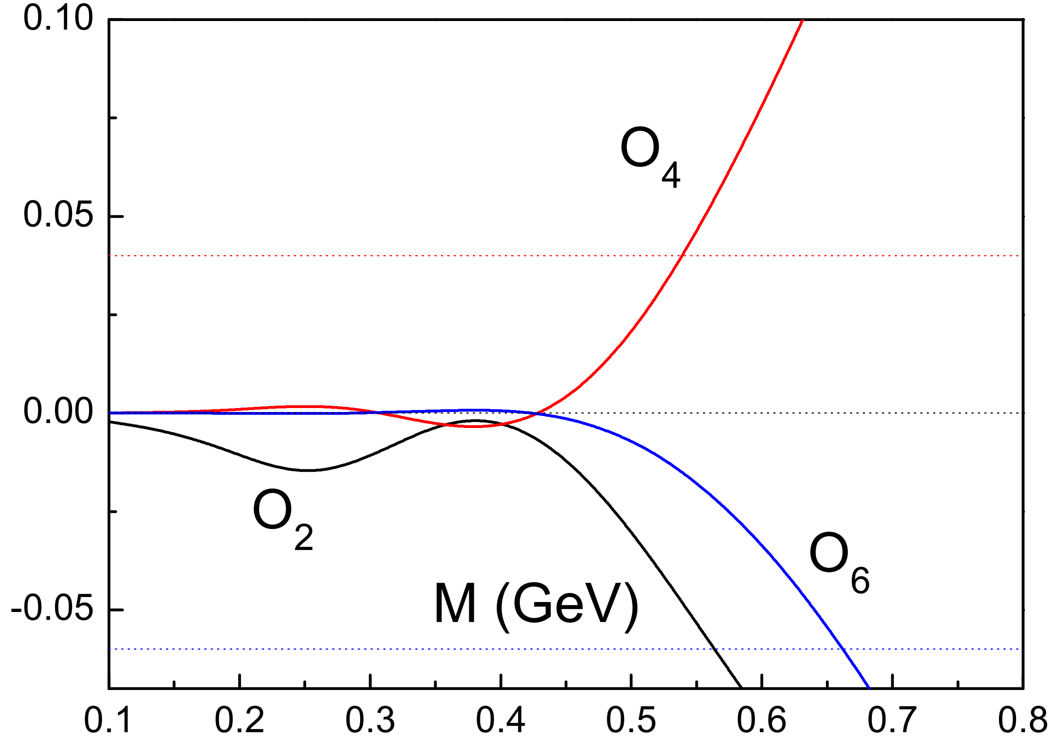
$$\mu_n(M^2) = \int_0^{s_0} ds\, s^{n-1} \exp(-s/M^2)\, \Delta R(s).$$

$$\tilde{\mu}(M^2) = \int_0^{s_0} ds\, \frac{1 - \exp(-s/M^2)}{s/M^2}\, \Delta R(s).$$

$$\langle O_2 \rangle = -2\, \mu_1(M^2) - \frac{\mu_2(M^2)}{2M^2} + 3\, \tilde{\mu}(M^2),$$

$$\langle O_4 \rangle = 6\, M^2 \mu_1(M^2) + 2\, \mu_2(M^2) - 6\, M^2 \tilde{\mu}(M^2),$$

$$\langle O_6 \rangle = -6\, M^4 \mu_1(M^2) - 3\, M^2 \mu_2(M^2) + 6\, M^4 \tilde{\mu}(M^2).$$



Optimal values of $\langle O_{2n} \rangle$ are compatible with zero

Summary

The theoretical method based on the nonperturbative VPT-expansion and the new expression for $R(s)$ has been used to describe

- $R_\tau^{V/A}$ – inclusive τ-decay characteristic;
- D – light Adler function;
- smeared R_Δ-function;
- $a_\mu^{\rm had}$ – hadronic contribution to the anomalous magnetic moment of the muon;
- $\Delta\alpha_{\rm had}^{(5)}$ – hadronic contribution to the fine structure constant.

The non-standard approach suggested works well and gives stable results down to low energy scale.

There is still room for improvements in theoretical methods.

The End